Seat No.: ________ Enrolment No.___________

# GUJARAT TECHNOLOGICAL UNIVERSITY

**BE - SEMESTER–III (NEW) EXAMINATION – WINTER 2021**

**Subject Code:3130005 Date:17-02-2022**

**Subject Name:Complex Variables and Partial Differential Equations**

**Time:10:30 AM TO 01:00 PM Total Marks:70**

**Instructions:**

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**
4. **Simple and non-programmable scientific calculators are allowed.**

| | | | Marks |
|---|---|---|---|
| **Q.1** | **(a)** | Represent $z = 7i$ into polar form and find the argument of $z$ and the principal value of the argument of $z$. | **03** |
| | **(b)** | State De Moivre's theorem. Find and plot all roots of $(1+i)^{\frac{1}{3}}$ in the complex plane. | **04** |
| | **(c)** | Using the method of separation of variables, solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ *when* $x = 0$. | **07** |
| **Q.2** | **(a)** | Define an analytic function. Write the necessary and sufficient condition for function $f(z)$ to be analytic. Show that $f(z) = \lvert z\rvert^2$ is nowhere analytic. | **03** |
| | **(b)** | Define the Mobious transformation. Determine the bilinear transformation which mapping the points $0, \infty, i$ *onto* $\infty, 2, 0$. | **04** |
| | **(c)** | **Attempt the following.** | |
| | | (i) Define the harmonic function. Show that $u = x^2 - y^2 + x$ is harmonic and find harmonic conjugate of $u$. | **04** |
| | | (ii) Show that $f(z) = \begin{cases} \frac{\text{Im}(z)}{\lvert z\rvert} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$ is not continuous at $z = 0$ | **03** |
| | | **OR** | |
| | **(c)** | **Attempt the following.** | |
| | | (i) Prove that $\cos^{-1} z = -i\ln(z + i\sqrt{1-z^2})$. | **4** |
| | | (ii) Find the values of $\text{Re} f(z)$ and $\text{Im} f(z)$ at the point $7+2i$, where $f(z) = \frac{1}{1-z}$. | **3** |
| **Q.3** | **(a)** | Evaluate $\int_C \bar{z}\,dz$, where $C$ is the right- half of the circle $\lvert z\rvert = 2$ and hence show that $\int_C \frac{dz}{z} = \pi i$. | **03** |
| | **(b)** | Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ and write the Maclaurin series for $e^{-z}$. | **04** |
| | **(c)** | Write the Cauchy integral theorem and Cauchy integral formula and hence evaluate: (i) $\oint_C \frac{e^z}{(z-1)(z-3)} dz; C : \lvert z\rvert = 2.$ (ii) $\oint_C e^z dz; C : \lvert z\rvert = 3$. | **07** |

**OR**

**Q.3** **(a)** Evaluate $\oint_C \frac{e^z}{z+i} dz$, where $c : |z-1| = 1$. **03**

**(b)** Develop the following functions into Maclaurin series:(i) $\cos^2 z$ (ii) $e^z \cos z$. **04**

**(c)** Evaluate $\int_C \text{Re}(z^2) dz$, where $C$ is the boundary of the square with vertices $0, i, 1+i, 1$ in the clockwise direction. **07**

**Q.4** **(a)** Define the singular points of $f(z)$. Find the singularity of $f(z)$ and classify as pole, essential singularity or removable singularity. where $f(z) = \frac{1-e^z}{z}$. **03**

**(b)** State the Cauchy residue theorem. Find the residue at its poles of $f(z) = \frac{z^2}{(z-1)^2 (z+2)}$ and hence evaluate $\oint_C f(z) dz$, $C : |z| = 3$. **04**

**(c)** Determine the Laurent series expansion of $f(z) = \frac{1}{(z+2)(z+4)}$ valid for regions **07**

$(i)$ $|z| < 2$

$(ii)$ $2 < |z| < 4$

$(iii)$ $|z| > 4$

**OR**

**Q.4** **(a)** Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$. **03**

**(b)** Form a partial differential equation by eliminating the arbitrary functions form the equations $z = f(x+ay) + \phi(x-ay)$. **04**

**(c)** Show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$. **07**

**Q.5** **(a)** Form a partial differential equation by eliminating the arbitrary function form $u = f(\frac{x}{y})$. **03**

**(b)** State the Lagrange's linear partial differential equation of first order and hence solve $x(y-z)p + y(z-x) = z(x-y)$ **04**

**(c)** Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ where $u(0, y) = 8e^{-3y}$. **07**

**OR**

**Q.5** **(a)** Obtain the general solution of $p + q^2 = 1$. **03**

**(b)** Solve by Charpit's method: $px + qy = pq$. **04**

**(c)** Find the solution of the wave equation $u_{tt} = c^2 u_{xx}$, $0 \le x \le L$ satisfying the condition $u(0,t) = u(L,t) = 0$, $u_t(x,0) = 0$, $u(x,0) = \frac{\pi x}{L}$, $0 \le x \le L$. **07**

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